

THE BURNSIDE RING-VALUED MORSE FORMULA FOR VECTOR FIELDS ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let G be a compact Lie group and $A(G)$ its Burnside Ring. For a compact smooth n -dimensional G -manifold X equipped with a generic G -invariant vector field v , we prove an equivariant analog of the Morse formula

$$Ind^G(v) = \sum_{k=0}^n (-1)^k \chi^G(\partial_k^+ X)$$

which takes its values in $A(G)$. Here $Ind^G(v)$ denotes the equivariant index of the field v , $\{\partial_k^+ X\}$ the v -induced Morse stratification (see [M]) of the boundary ∂X , and $\chi^G(\partial_k^+ X)$ the class of the $(n-k)$ -manifold $\partial_k^+ X$ in $A(G)$.

We examine some applications of this formula to the equivariant real algebraic fields v in compact domains $X \subset \mathbb{R}^n$ defined via a generic polynomial inequality. Next, we link the above formula with the equivariant degrees of certain Gauss maps. This link is an equivariant generalization of Gottlieb's formulas ([G], [G1]).

1. INTRODUCTION

Let X be a compact smooth n -manifold with boundary ∂X . A *generic* (see Definition 1.1) vector field v on X which is nonzero along ∂X gives rise to a stratification

$$(1.1) \quad X := \partial_0^+ X \supset \partial_1^+ X \supset \partial_2^+ X \supset \cdots \supset \partial_n^+ X$$

by compact submanifolds, where $\dim(\partial_j^+ X) = n - j$. Here $\partial_1^+ X$ is the part of the boundary $\partial_1 X := \partial X$ where v points inside X . By definition, $\partial_2 X$ is the locus where v is *tangent* to the boundary $\partial_1 X$. Its portion $\partial_2^+ X \subset \partial_2 X$ consists of points where v points inside $\partial_1^+ X$. Similarly, $\partial_3 X$ is the locus where v is tangent to $\partial_2 X$. In the same spirit, $\partial_3^+ X \subset \partial_3 X$ consists of points where v points inside $\partial_2^+ X$. Continuing this process, we get the Morse stratification (1.1).

In Section 2, Theorem 2.1, we investigate quite strong restrictions on the nature of the stratification $\{\partial_j X\}$ (and thus of $\{\partial_j^+ X\}$) imposed by a G -symmetry of the field v and the manifold X .

Definition 1.1. *We say that a field v is generic, if for each k , viewed as a section of the bundle $T(\partial_k X)|_{\partial_{k+1} X}$, v is transversal to the zero section of the tangent bundle $T(\partial_{k+1} X)$.*

In his groundbreaking 1929 paper [Mo], Morse discovered a beautiful connection between stratification (1.1) and the index $Ind(v)$ of the field v . It is expressed in

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terms of the Euler numbers of the strata from (1.1):

$$(1.2) \quad \text{Ind}(v) = \sum_{k=0}^n (-1)^k \chi(\partial_k^+ X).$$

Our main observation is that, for any compact Lie group G and a generic G -equivariant vector field v , a similar formula with values in the Burnside ring $A(G)$ holds. Theorem 3.1 is an equivariant version of the Morse formula (1.2) for generic symmetric vector fields on manifolds with boundary. It is a generalization of Theorem 6.6, [LR], for fields that do not necessarily point outward X along its boundary¹.

In Section 4, Theorem 4.2, we combine some results of Khovanskii [Kh] about (non-equivariant) indices of real algebraic vector fields in polynomially-defined domains in \mathbb{R}^n with formulas (3.9) and (3.10) from Theorem 3.1 to get a handle on the size of equivariant indices of such fields.

In Section 5, Theorem 5.1, we obtain an $A(G)$ -valued version of Gottlieb's "Topological Gauss-Bonnet Theorem" [G], [G1]. These results connect the indices of pullback fields F^*w under smooth G -maps $F : X \rightarrow V$ to the equivariant degree $\text{Deg}^G(\gamma)$ of the Gauss map $\gamma : \partial X \rightarrow S(V)$. Here w is a nonsingular G -invariant field on a space of a G -representation V , and $F|_{\partial X}$ is an immersion.

In Section 6, we speculate about some generalizations of our equivariant index formulas, the generalizations which reside in refined versions of the ring $A(G)$.

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2. G -INVARIANT VECTOR FIELDS AND MORSE STRATIFICATIONS

When v is both generic and G -invariant, the Morse stratification $\{\partial_k^+ X\}$ is invariant as well. In fact, it is quite special. For example, the following proposition is valid:

Theorem 2.1. *Let a compact Lie group G act faithfully on a smooth compact n -manifold X with an oriented boundary $\partial_1 X$. Then, for any k and a generic G -equivariant vector field v ,*

- *the main orbit-type of $\partial_k X$ is G .*
- *for all $k > n - \dim(G)$, we have $\partial_k X = \emptyset$.*
- *the sets $\partial_{n-\dim(G)}^\pm X$ are disjoint unions of free G -orbits.*
- *if G is connected, the sets $\partial_{n-1-\dim(G)}^\pm X$ are disjoint unions of the following G -spaces:*
 - (1) $G \times [0, 1]$,
 - (2) $G \times S^1$,
 - (3) *the mapping cones $\mathcal{C}(G \rightarrow G/H)$, where $H \approx SO(2)$ or $SU(2)$,*
 - (4) $\mathcal{C}(G \rightarrow G/H) \cup_G \mathcal{C}(G \rightarrow G/K)$, *where $H, K \approx SO(2)$ or $SU(2)$.*

¹Actually, [LR] deals with the category of cocompact discrete G -actions.

²That is, (X, v) is $(n + 1 - \dim(G))$ -convex in the terminology of [K].

Proof. Pick $x \in \partial_k X^\circ := \partial_k X \setminus \partial_{k+1} X$ and let $H = G_x$. Because v is H -invariant, a linearization of the H -action at x must leave the flag F_x of tangent half-spaces $\{T_x^+(\partial_l^+ X)\}_{l \leq k}$ invariant as well. Pick a G -invariant metric on X . Consider the unique "frame" ν_x comprised of k mutually orthogonal rays r_1, \dots, r_k generated by intersecting the flag F_x with the space N_x^k , normal in $T_x(X)$ to $T_x(\partial_k X)$. Specifically, for all $l < k$, the positive cone $\text{Span}^+\{r_1, \dots, r_l\} = N_x^k \cap T_x^+(\partial_{k-l}^+ X)$. The "frame" ν_x must be H -invariant. Thus, in a basis consistent with ν_x , the H -action is diagonal with the positive eigenvalues. Since H is compact, its action on the space N_x^k (spanned by ν_x) is trivial. At the same time, the H -action on the space tangent to the trajectory Gx at x is trivial as well.

Now consider the 0-dimensional G -invariant sets $\partial_n^\pm X$. By the argument above, for any $x \in \partial_n^\pm X$, the group $H = G_x$ acts trivially on the n -flag F_x , and thus, on $T_x X$. Since the G -action on X is faithful, we get $H = 1$. Hence, either $\partial_n^\pm X = \emptyset$ when $\dim(G) > 0$, or $\dim(G) = 0$ and $\partial_n^\pm X$ is a union of free orbits.

Next, consider the case $\dim(G) \geq 1$ and focus on the 1-dimensional G -invariant sets $\partial_{n-1}^\pm X$. By the argument above, for any $x \in \partial_{n-1}^\pm X$, the group $H = G_x$ acts trivially on the vector space N_x^{n-1} . In addition, H acts trivially on the 1-dimensional space tangent to the orbit G/H through x . Since $\partial_{n-1} X$ is 1-dimensional, the space tangent to the orbit and the space tangent to $\partial_{n-1} X$ coincide. As a result, the H action is trivial on $T_x X$, and thus, in the vicinity of x . Since the G -action on X is faithful, we get again $H = 1$. Hence, either $\dim(G) > 1$ and $\partial_{n-1}^\pm X = \emptyset$, or $\dim(G) = 1$ and $\partial_{n-1}^\pm X$ is a union of free G -orbits.

Finally, let us treat the general case. For $k > 0$ and a generic G -invariant v , $\partial_k X$ is a closed G -manifold. Pick a main orbit $G/H \subset \partial_k X$, $H = G_x$ for some $x \in \partial_k X$. According to [B], an invariant regular neighborhood of $G/H \subset \partial_k X$ is diffeomorphic to the balanced product $G \times_H V$, where V is a space of an H -representation ψ . If this ψ is a non-trivial representation, G/H is not the main orbit: the stationary groups of some points from $G \times_H (V \setminus \{0\})$ must be smaller than H . Thus, we can assume that H acts trivially on V . In addition, it acts trivially on the tangent space to the orbit G/G_x at x , as well as on the normal space N_x^k . As a result, the G_x -action on $T_x X$ is trivial. Because G acts faithfully on X , $G_x = 1$; that is, the main orbit-type of $\partial_k X$ (and hence of $\partial_k^\pm X$) is G ! Therefore, when $n - k < \dim(G)$, $\partial_k X = \emptyset$.

When $n - k = \dim(G)$, each main orbit G is open and dense in some G -invariant union of connected components of $\partial_k X$. Since both G and $\partial_k X$ are closed manifolds, $\partial_k X$ is G -diffeomorphic to a disjoint union of several G 's.

When $n - k = \dim(G) + 1$ and G is connected, it is present as a codimension one main orbit-type of each connected component of $\partial_k X$. Fortunately, connected closed G -manifolds of this kind are very rigid: they all are either products $G \times S^1$, or unions of two mapping cylinders $\mathcal{C}(G \rightarrow G/H)$, $\mathcal{C}(G \rightarrow G/K)$ with H and K being diffeomorphic to a sphere [B], [K1]. The two mapping cylinders share the same "top" G . The only spheres among the Lie groups are $O(1) \approx \mathbb{Z}_2$, $SO(2)$, $SU(2)$. However, the \mathbb{Z}_2 -option must be excluded: it leads to a non-orientable manifold, while all $\partial_k X$ are oriented for $k > 0$. This leaves us with the models

$$\mathcal{C}(G \rightarrow G/H) \cup_G \mathcal{C}(G \rightarrow G/K),$$

where $H, K = SO(2), SU(2)$, for the components of $\partial_{n-1-\dim(G)} X$.

Similarly, the models for $\partial_{n-1-\dim(G)}^\pm X$ are: $\mathcal{C}(G \rightarrow G/H) \cup_G \mathcal{C}(G \rightarrow G/K)$, $\mathcal{C}(G \rightarrow G/H)$, $G \times S^1$, and $G \times [0, 1]$, where $H, K = SO(2), SU(2)$. \square

Corollary 2.1. *Let X, v be as in Theorem 2.1. Assume that G is finite. Then, for some integer $l \geq 0$, $|\partial_n X| = 2l|G|$, provided $|G|$ being odd, and $|\partial_n X| = l|G|$, provided $|G|$ being even.*

If $G = SO(2)$, then $\partial_n X = \emptyset$, $\partial_{n-1}^\pm X$ each is a union of circles on which G acts freely, and each $\partial_{n-2}^\pm X$ is a union of tori $SO(2) \times S^1$, cylinders $SO(2) \times [0, 1]$, 2-disks, and 2-spheres. The $SO(2)$ -action on each disk or sphere is semi-free.

Question. For a given compact G -manifold X of dimension n with orientable boundary, what is the minimal number of free trajectories that form the sets $\partial_{n-\dim(G)}(X)$ and $\partial_{n-\dim(G)}^\pm(X)$? The minimum is taken over the space of all generic G -invariant vector fields v^3 .

Lemma 2.1. *For a G -invariant field v and each compact subgroup $H \subset G$, we have $\partial_k^+(X^H, v|_{X^H}) = X^H \cap \partial_k^+(X, v)$.*

Proof. A key observation here is that, for each compact subgroup $H \subset G$, any G -invariant field v on X is tangent to the submanifold $X^H \subset X$. Indeed, in a G -invariant metric g , the normal to X^H component ν_H of v must be H -invariant, and thus vanishes for $H \neq 1$. Each submanifold X^H meets $\partial_1 X$ transversally: just consider a linearization of the G_x -action, $G_x \supset H$, at a typical point $x \in X^H \cap \partial_1 X$. Since v is tangent to X^H , evidently, $\partial_1^+(X^H, v|_{X^H}) = X^H \cap \partial_1^+(X, v)$.

We proceed by induction on k . Assume that X^H and $\partial_s X$ are transversal, $\partial_s(X^H, v|_{X^H}) = X^H \cap \partial_s(X, v)$, and $\partial_s^+(X^H, v|_{X^H}) = X^H \cap \partial_s^+(X, v)$ for all $s < k$. If $x \in X^H \cap \partial_k^+(X, v)$ and $v(x)$ is tangent to $\partial_{k-1} X \cap X^H = \partial_{k-1} X^H$, then $v(x)$ points inside of $\partial_{k-1}^+ X$ if and only if it points inside $\partial_{k-1}^+ X^H$. Thus, $\partial_k(X^H, v|_{X^H}) = X^H \cap \partial_k(X, v)$, which completes the induction step. \square

3. EQUIVARIANT MORSE FORMULA FOR VECTOR FIELDS

Following tom Dieck, consider a ring $A(G)$ generated over \mathbb{Z} by equivalence classes of compact differentiable G -manifolds. Two G -manifolds X and Y are said to be equivalent, if for any compact subgroup $H \subset G$, the Euler numbers $\chi(X^H)$ and $\chi(Y^H)$ are equal. This equivalence relation respects disjoint unions and cartesian products of G -spaces. Therefore, $A(G)$ is a ring with the sum and product operations induced by disjoint unions and cartesian product of equivalence classes of G -manifolds.

In particular, any compact G -manifold X gives rise to an element $\chi^G(X) \in A(G)$.

Let $ch : A(G) \rightarrow \prod_{(H) \in conj(G)} \mathbb{Z}$ be a ring homomorphism induced by the correspondence $\chi^G(X) \mapsto \{\chi(X^H)\}_{(H) \in conj(G)}$, where $conj(G)$ denotes the conjugacy classes of compact subgroups in G . It turns out that ch is a *monomorphism* [D]. We denote by ch_H the (H) -indexed component of ch .

Consider only the conjugacy classes (H) with the finite quotients $WH := NH/H$ (NH standing for the normalizer of H in G) and denote by $\Phi(G)$ the set of such (H) . Next, form a free abelian group $A'(G)$ generated by elements of $\Phi(G)$ (equivalently, by the orbit-types $\{G/H\}_{(H) \in \Phi(G)}$ regarded as G -spaces). By [D], Theorem 1, the

³In [K] we proved that, for $n = 3$ and $G = 1$, this minimum is zero.

natural homomorphism $A'(G) \rightarrow A(G)$ is an isomorphism. As a result, any element $[X] \in A(G)$ is detected by $\{ch_H([X])\}_{(H) \in \Phi(G)}$ alone.

Let U_Z be a G -invariant regular neighborhood⁴ of the zero set

$$Z := Z(v) = \{x \in X \mid v(x) = 0\}$$

and a smooth manifold. Assume that v is in general position with respect to $\partial_1 U_Z := \partial U_Z$. Note that, for each $H \subset G$, U_Z^H and Z^H are homotopy equivalent.

Inspired by [M], we propose the following

Definition 3.1.

$$(3.1) \quad ind^G(v) := \sum_{k=0}^n (-1)^k \chi^G(\partial_k^+ U_Z) = \chi^G(Z) + \sum_{k=1}^n (-1)^k \chi^G(\partial_k^+ U_Z)$$

Lemma 3.1. *For each $(H) \in conj(G)$, $ch_H(ind^G(v)) = ind(v|_{X^H})$, the non-equivariant index of the field v on X^H (equivalently, on U_Z^H).*

Proof. By Lemma 2.1, v is tangent to $X^H \supset U_Z^H$ and $(\partial_k^+ U_Z)^H = \partial_k^+(U_Z^H)$. Since v is tangent to X^H , all the zeros of $v|_{X^H}$ are among the zeros of v on X , and no new zeros appear. By the Morse formula (1.2) ([M]),

$$ind(v|_{U_Z^H}) = \sum_{k=0}^n (-1)^k \chi(\partial_k^+(U_Z^H)) = \sum_{k=0}^n (-1)^k \chi(\partial_k^+(U_Z)^H).$$

Thus,

$$(3.2) \quad ch_H(ind^G(v)) = \sum_{k=0}^n (-1)^k ch_H(\partial_k^+ U_Z) = ind(v|_{U_Z^H}) = ind(v|_{X^H}).$$

□

For G -invariant vector fields, there is an alternative definition of the equivariant index. It is more involved and mimics the classical definition of index as the sum of local degrees produced by the field v in the vicinity of its zero set. Let us describe this definition along the lines of [L], [LR].

The tangent space $T_{(x,0)}(TX)$ decomposes as a direct sum of the subspaces $T_x X \oplus T_x X$, where the first factor is thought as being tangent to the zero section $\epsilon : X \rightarrow TX$ and the second one to the fiber of the bundle $TX \rightarrow X$. Let $\alpha : T_x X \oplus T_x X \rightarrow T_{(x,0)}(TX)$ denote this G_x -equivariant isomorphism. Consider a G -equivariant vector field v on X , viewed as a section $v : X \rightarrow TX$ of the tangent G -bundle TX . Assume that v is transversal to the zero section ϵ (due to the "doubling" nature of the G_x -space $T_{(x,0)}(TX)$, this equivariant transversality of v and ϵ is available by a general position argument). If $v(x) = 0$ for some $x \in X$, then $T_{(x,0)}(TX)$ decomposes a direct sum of $Dv(T_x X)$ and $D\epsilon(T_x X)$ as well. Here Dv and $D\epsilon$ denote the differentials of the respective maps. Consider the G_x -equivariant isomorphism

$$(3.3) \quad \mathcal{D}_{x,v} : T_x X \xrightarrow{Dv} T_{(x,0)}(TX) \xrightarrow{\alpha^{-1}} T_x X \oplus T_x X \xrightarrow{pr_2} T_x X$$

, where pr_2 stands for the projection on the second factor. $\mathcal{D}_{x,v}$ induces a G_x -map $\mathcal{D}_{x,v}^c : T_x X^c \rightarrow T_x X^c$ on the one point compactification $T_x X^c$ of $T_x X$. For such

⁴In particular, Z is an equivariant deformation retract of U_Z

a map, its degree, $Deg^{G_x}(\mathcal{D}_{x,v}^c) \in A'(G_x) \approx A(G_x)$, is defined in terms of the equivariant Lefschetz classes $\Lambda^{G_x}(\mathcal{D}_{x,v}^c)$, $\Lambda^{G_x}(Id_{T_x X^c})$ by the formula

$$(3.4) \quad Deg^{G_x}(\mathcal{D}_{x,v}^c) = [\Lambda^{G_x}(\mathcal{D}_{x,v}^c) - 1][\Lambda^{G_x}(Id_{T_x X^c}) - 1]$$

(see [L], [LR]), where $1 \in A(G_x)$ stands for the class of a point.

In Section 4, we will return briefly to the definition of an equivariant degree $Deg^G(f)$ for a general G -map $f : X \rightarrow Y$ of two G -manifolds.

Meanwhile, let us recall the definition of the equivariant Lefschetz class $\Lambda^G(f) \in A'(G)$ of a G -map $f : X \rightarrow X$, X being a finite G -CW-complex⁵. In particular, for each $(H) \in \Phi(G)$, the orbit-space $WH \setminus X^H$ has the structure of a finite CW -complex. Let

$$(3.5) \quad \Lambda^G(f) := \sum_{(H) \in \Phi(G)} L^{\mathbb{Z}[WH]}(f^H, f^{>H}) \cdot \chi^G(G/H)$$

, where $\chi^G(G/H)$ is the class of the homogeneous space G/H in $A'(G)$, and the integer

$$(3.6) \quad L^{\mathbb{Z}[WH]}(f^H, f^{>H}) = \sum_{p \geq 0} (-1)^p \cdot tr_{\mathbb{Z}[WH]}(C_p(f^H, f^{>H})).$$

In (3.6), we employ the \mathbb{Z} -valued trace $tr_{\mathbb{Z}[WH]}$ of the f -induced automorphism $C_p(f^H, f^{>H})$ of the finitely generated free $\mathbb{Z}[WH]$ -module $C_p(X^H, X^{>H}; \mathbb{Z})$, the module of the relative cellular p -chains of the pair $(X^H, X^{>H})$ (see [LR] for details).

Lemma 3.2. *The equivariant Lefschetz class $\Lambda^G(f)$ from (3.5) is detected by the non-equivariant Lefschetz numbers $\{\Lambda(f^H) \in \mathbb{Z}\}_{(H) \in \Phi(G)}$.*

Proof. By [D], Theorem 2, only the orbit-types G/H with WH being finite contribute to the homomorphism $ch : A(G) \rightarrow \prod_{(H)} \mathbb{Z}$. Therefore, we will consider only the orbit-types from $\Phi(G)$.

In (3.5), we took into account only the orbit-types $(H) \in \Phi(G)$ with non-vanishing coefficients $L^{\mathbb{Z}[WH]}(f^H, f^{>H})$ (defined by (3.6)). Among them, pick a minimal⁶ orbit-type (K) and apply ch_K to $\Lambda^G(f)$. Then,

$$\begin{aligned} ch_K(\Lambda^G(f)) &= \sum_{(H) \in \Phi(G)} \left\{ \sum_{p \geq 0} (-1)^p tr_{\mathbb{Z}[WH]}(C_p(f^H, f^{>H})) \right\} \cdot \chi(G/H^K) \\ &= \left\{ \sum_{p \geq 0} (-1)^p tr_{\mathbb{Z}[WK]}(C_p(f^K, f^{>K})) \right\} \cdot \chi(G/K^K) = \\ &= \left\{ \sum_{p \geq 0} (-1)^p tr_{\mathbb{Z}[WK]}(C_p(f^K, f^{>K})) \right\} \cdot |WK| = \\ (3.7) \quad &= \sum_{p \geq 0} (-1)^p tr_{\mathbb{Z}}(C_p(f^K, f^{>K})) := \Lambda(f^K). \end{aligned}$$

In other words, for such a minimal $(K) \in \Phi(G)$,

$$L^{\mathbb{Z}[WK]}(f^K, f^{>K}) = |WK|^{-1} \Lambda(f^K) = |WK|^{-1} ch_K(\Lambda^G(f)).$$

⁵for example, see [O] for the definition of a G -CW-complex

⁶i.e. a maximal subgroup $K \subset G$

The rest of the argument is performed inductively, the induction step being applied to a minimal orbit-type (K') in $\Phi(G) \setminus (K)$ with a non-zero coefficient $L^{\mathbb{Z}[WK']}(f^{K'}, f^{>K'})$. Indeed, compute $ch_{K'}$ of

$$\beta := \Lambda^G(f) - L^{\mathbb{Z}[WK]}(f^K, f^{>K}) \cdot \chi^G(G/K)$$

to conclude, in a similar way, that $ch_{K'}(\beta)$ equals $\Lambda(f^{K'})$. \square

Corollary 3.1. *$Deg^{G_x}(\mathcal{D}_{x,v}^c) \in A(G_x)$ is determined by the \mathbb{Z} -valued degrees $\{Deg(\mathcal{D}_{x,v}^{H,c})\}_{(H) \in \Phi(G_x)}$ of the maps $\mathcal{D}_{x,v}^{H,c} := \mathcal{D}_{x,v}^c| : (T_x X^c)^H \rightarrow (T_x X^c)^H$. For all $H \notin \Phi(G_x)$, $H \subset G_x$, $ch_H[Deg^{G_x}(\mathcal{D}_{x,v}^c)] = 0$.*

Proof. Apply the arguments (3.7) of Lemma 3.2 (with G being replaced by G_x) to formula (3.4). \square

Now put

$$(3.8) \quad Ind^G(v) := \sum_{G_x \in G \setminus Z(v)} G \times_{G_x} [Deg^{G_x}(\mathcal{D}_{x,v}^c)]$$

Lemma 3.3. *$Ind^G(v)$, defined by (3.8), and $ind^G(v)$, defined by (3.1), are equal.*

Proof. For each x from the orbit-space $G \setminus Z(v)$, consider a small G_x -equivariant n -disk $D_x \subset X$ centered on x . Note that, when $Z(v)$ is discrete, the quotient G/G_x must be finite. Evidently, $\partial_k^+ U_Z \approx \coprod_x G \times_{G_x} (\partial_k^+ D_x)$. Therefore, in order to prove $Ind^G(v) = ind^G(v)$, it will suffice to verify that, for each $x \in G \setminus Z(v)$,

$$Deg^{G_x}(\mathcal{D}_{x,v}^c) = \sum_k (-1)^k \chi^{G_x}(\partial_k^+ D_x)$$

in $A'(G_x)$. The last equality follows from Corollary 3.1 together with the validity of non-equivariant Morse formulas of the type (1.2) (with G being replaced by $H \subset G_x$ and X by D_x). \square

Theorem 3.1. *Let G be a compact Lie group, and let X ⁷ be a compact smooth n -dimensional G -manifold with orientable boundary $\partial_1 X$. Let v be a generic G -invariant vector field on X (with isolated singularities). Then the equivariant Morse formula*

$$(3.9) \quad Ind^G(v) = ind^G(v) = \sum_{k=0}^n (-1)^k \chi^G(\partial_k^+ X)$$

is valid in the ring $A(G)$. If the G -action on X is faithful and $\dim(G) > 0$,

$$(3.10) \quad Ind^G(v) = ind^G(v) = \sum_{k=0}^{n-1-\dim(G)} (-1)^k \chi^G(\partial_k^+ X)$$

When G is connected, the contribution of the stratum $\partial_{n-1-\dim(G)}^+ X$ to $Ind^G(v)$ is quite special:

$$\chi^G(\partial_{n-1-\dim(G)}^+ X) = \sum_{\{(H) \in \Phi(G) \mid H \approx SO(2), SU(2)\}} n_H \cdot \chi^G(G/H),$$

⁷also denoted by $\partial_0^+ X$

where $\{n_H\}$ are some nonnegative integers. In particular, if a connected G is such that, for each $H \subset G$ isomorphic to $SO(2)$ or $SU(2)$, the group WH is infinite, then the contribution of the stratum $\partial_{n-1-\dim(G)}^+ X$ vanishes.

Proof. By Lemma 3.3, $Ind^G(v) = ind^G(v)$. Since the elements of $A(G)$ are detected by the character map $ch : A(G) \rightarrow \prod_{(H) \in \Phi(G)} \mathbb{Z}$, it will suffice to check the validity of (3.9) by applying ch to both sides of the conjectured equation $ind^G(v) = \sum_{k=0}^n (-1)^k \chi^G(\partial_k^+ X)$.

By Lemma 2.1, for any $H \subset G$, $ch_H[\chi^G(\partial_k^+ X)] := \chi((\partial_k^+ X)^H) = \chi(\partial_k^+(X^H))$. Thus,

$$(3.11) \quad ch_H\left(\sum_{k=0}^n (-1)^k \chi^G(\partial_k^+ X)\right) = \sum_{k=0}^n (-1)^k \chi(\partial_k^+(X^H))$$

On the other hand, by Lemma 3.1 and formula (3.2), $ch_H(ind^G(v)) = ind(v|_{X^H})$. Since the Morse formula claims that $ind(v|_{X^H}) = \sum_{k=0}^n (-1)^k \chi(\partial_k^+(X^H))$, we get the desired equality $ch_H(ind^G(v)) = ch_H(\sum_{k=0}^n (-1)^k \chi^G(\partial_k^+ X))$ for all $(H) \in conj(G)$.

In order to derive formula (3.10) from formula (3.9), we employ Theorem 2.1 to conclude that $\partial_k^+ X = \emptyset$ for all $k > n - \dim(G)$. Moreover, $\partial_{n-\dim(G)}^+ X$ is the union of free G -orbits, and thus $\chi^G(\partial_{n-\dim(G)}^+ X) = 0$, provided $\dim(G) > 0$.

The last claim of the theorem can be validated by the following observation. Models (1) and (2) from Theorem 2.1 have zero classes in $A(G)$, model (3) is G -homotopy equivalent to G/H , and model (4), by the additivity of Euler characteristics, produces the element $\chi^G(G/H) + \chi^G(G/K)$.

If a connected G is such that, for each $H \subset G$ isomorphic to $SO(2)$ or $SU(2)$, WH is infinite, the space G/H admits a free S^1 action, $S^1 \subset WH$. Thus, $\chi^G(G/H) = 0$. As a result, for such a G , we get a simplification of (3.10):

$$(3.12) \quad Ind^G(v) = \sum_{k=0}^{n-2-\dim(G)} (-1)^k \chi^G(\partial_k^+ X).$$

□

Corollary 3.2. *Let X and v be as in Theorem 3.1. Denote by v_1^+ an orthogonal projection (with respect to a G -invariant metric on X) of the field $v|_{\partial_1^+ X}$ on the tangent space $T(\partial_1^+ X)$. Assume that v_1^+ has only isolated singularities. Then the following formula holds in $A(G)$:*

$$(3.13) \quad \chi^G(X) = ind^G(v) + ind^G(v_1^+)$$

Proof. In view of the recursive nature of the Morse stratification $\{\partial_k^+ X\}$, (3.9) and (3.13), applied to all consecutive terms in the stratification, are equivalent formulas. □

4. POLYNOMIAL VECTOR FIELDS IN POLYNOMIAL DOMAINS

Now, we turn our attention to polynomial vector fields v in domains in \mathbb{R}^n that are defined by polynomial inequalities.

Consider a real n -dimensional vector space W and a G -representation $\Psi : G \rightarrow GL_{\mathbb{R}}(W)$. Denote by \mathcal{P}_{Ψ} the algebra of invariant polynomials on W . Let v be

a G -invariant vector field in $W \approx \mathbb{R}^n$ which has polynomial components $\{P_i \in \mathcal{P}_\Psi\}_{1 \leq i \leq n}$. Assume that $\deg(P_i) \leq m_i$. We denote by \tilde{P}_i the homogenized versions of P_i , that is, $\tilde{P}_i(1, x_1, \dots, x_n) = P_i(x_1, \dots, x_n)$.

Also consider an invariant polynomial $Q \in \mathcal{P}_\Psi$ of degree d . We say that the pair (v, Q) is *non-degenerated*, if 1) all zeros of v are simple, 2) the system $\{x_0 = 0, \tilde{P}_i(x_0, x_1, \dots, x_n) = 0\}_{1 \leq i \leq n}$ has only a trivial solution⁸, and 3) the hypersurface $Q = 0$ does not contain the zeros of v .

For any non-increasing sequence m_1, m_2, \dots, m_n of natural numbers, form the parallelepiped $\Pi(m_1, \dots, m_n)$ in \mathbb{R}^n defined by

$$(4.1) \quad \{0 \leq x_i \leq m_i - 1\}_{1 \leq i \leq n}$$

Let $O(d, m_1, \dots, m_n)$ be the number of integral lattice points (x_1, \dots, x_n) in $\Pi(m_1, \dots, m_n)$, subject to the inequalities

$$(4.2) \quad \frac{1}{2}(m_1 + \dots + m_n - d - n) \leq x_1 + \dots + x_n \leq \frac{1}{2}(m_1 + \dots + m_n - n)$$

Consider the domain $X_Q := \{w \in W \mid Q(w) \geq 0\}$. Then, according to a theorem of Khovanskii (Theorem 1, [Kh]), for a non-degenerated pair (v, Q) as above, the absolute value of index, $|\text{ind}(v)|$, in X_Q is bounded from above by $O(d, m_1, \dots, m_n)$; moreover, this estimate is sharp.

Assume that X_Q is compact with a smooth boundary $\partial_1 X_Q$ and a polynomial field v is generic (see Definition 1.1) in relation to the boundary. Combining the Khovanskii Theorem and Morse Formula (1.2), we get

Theorem 4.1. *For a polynomial field v in X_Q as above,*

$$(4.3) \quad \left| \sum_{k=0}^n (-1)^k \chi(\partial_k^+ X_Q) \right| \leq O(d, m_1, \dots, m_n)$$

Moreover, there exist (Q, v) for which the inequality (4.3) can be replaced by the equality.

Let $V \subset \mathbb{R}^n$ be a vector subspace of dimension l . Let $d_V \leq d$ denote the degree of the polynomial Q being restricted to the subspace V . Consider a subspace $U \subset \mathbb{R}^n$ which is spanned by some set of l basic vectors $\{e_{j_1}, \dots, e_{j_l}\}$ in \mathbb{R}^n and such that the obvious orthogonal projection $p_U : \mathbb{R}^n \rightarrow U$, being restricted to V , is onto. Denote by $\mathcal{U}(V)$ the finite set of such U 's.

Put

$$(4.4) \quad O(V; d_V, m_1, \dots, m_n) = \min_{U \in \mathcal{U}(V)} \left\{ O(d_V, m_{j_1}, \dots, m_{j_l}) \right\},$$

where $U = \text{span}\{e_{j_1}, \dots, e_{j_l}\}$ and $O(d_V, m_{j_1}, \dots, m_{j_l})$ is defined as in (4.2).

The theorem below generalizes the estimates (4.3) for an equivariant setting.

Theorem 4.2. *Let G be a compact Lie group. Pick a sequence $m_1 \geq m_2 \geq \dots \geq m_n > 0$ of integers. Consider a representation $\Psi : G \rightarrow GL(n, \mathbb{R})$, an invariant polynomial $Q(x_1, \dots, x_n)$ of degree d , and a non-degenerate $\Psi(G)$ -invariant polynomial field*

$$v = (P_1(x_1, \dots, x_n), \dots, P_n(x_1, \dots, x_n))$$

⁸i.e. no zero of v escapes to infinity.

such that $\deg(P_i) \leq m_i$. Assume that $X_Q := \{\vec{x} \in \mathbb{R}^n \mid Q(\vec{x}) \geq 0\}$ is a compact⁹ domain with a smooth boundary $\partial_1 X_Q$ and that v is in general position with respect to $\partial_1 X_Q$.

Then the image $\prod_{(H) \in \Phi(G)} z_{(H)}$ of the element $\sum_{k=0}^n (-1)^k \chi(\partial_k^+ X_Q) \in A(G)$ under the character monomorphism $ch : A(G) \rightarrow \prod_{(H) \in \Phi(G)} \mathbb{Z}$ belongs to the parallelepiped $\mathcal{P}_{\Phi(G)}$ defined by the inequalities:

$$(4.5) \quad |z_{(H)}| \leq O((\mathbb{R}^n)^H; d_{(\mathbb{R}^n)^H}, m_1, \dots, m_n)$$

Proof. Let $l = \dim((\mathbb{R}^n)^H)$. For each $U = \text{span}\{e_{j_1}, \dots, e_{j_l}\}$ as above with the property $p_U : (\mathbb{R}^n)^H \rightarrow U$ being onto, the projection p_U induces an invertible linear transformation of the pairs (v, X_Q^H) and $(p_U(v), p_U(X_Q^H))$, where $X_Q^H := X_Q \cap (\mathbb{R}^n)^H$. Let I be the set of indices complementary to the set $J := \{j_1, \dots, j_l\}$. Then the components \tilde{P}_j of $p_U(v)$ are obtained by substituting $\{x_i = 0\}_{i \in I}$ into $\{P_j(x_1, \dots, x_n)\}_{j \in J}$. Note that $\deg(\tilde{P}_j) \leq \deg(P_j) \leq m_j$. Also, $p_U(X_Q^H) \subset \text{Span}\{e_j\}_{j \in J}$ is defined by a polynomial inequality $\tilde{Q}(x_{j_1}, \dots, x_{j_l}) \geq 0$ which is obtained from $Q(x_1, \dots, x_n) \geq 0$ by a substitution that expresses each $x_i, i \in I$, as linear combination of the $\{x_j\}_{j \in J}$. Evidently, $\deg(\tilde{Q}) = d_{(\mathbb{R}^n)^H} \leq d$. Since $v|_{(\mathbb{R}^n)^H}$ is parallel to $(\mathbb{R}^n)^H$, p_U also maps $\partial_k^+(X_Q^H)$ onto $\partial_k^+[p_U(X_Q^H)]$. By [Kh] and Theorem 3.1, $|Ind(p_U(v))|$ in $p_U(X_Q^H)$ (equivalently, $|\sum_{k=0}^n (-1)^k \chi(\partial_k^+ p_U(X_Q^H))|$) has an upper boundary $O(d_{(\mathbb{R}^n)^H}, m_{j_1}, \dots, m_{j_l})$. Via the linear diffeomorphism p_U , $Ind(p_U(v))$ in $\{\tilde{Q} \geq 0\}$ and $Ind(v|_{X_Q^H})$ in X_Q^H are equal. Thus, $|Ind(p_U(v))| \leq O(d_{(\mathbb{R}^n)^H}, m_{j_1}, \dots, m_{j_l})$. In view of formula (4.3) and Lemma 3.1,

$$|ch_H(Ind^G(v))| = |Ind(v|_{X_Q^H})| \leq O(d_{(\mathbb{R}^n)^H}, m_{j_1}, \dots, m_{j_l}),$$

and thus using formula-definition (4.4),

$$|ch_H(Ind^G(v))| \leq O((\mathbb{R}^n)^H; d_{(\mathbb{R}^n)^H}, m_1, \dots, m_n). \quad \square$$

Corollary 4.1. *Let v be as in Theorem 4.2 and Ψ be an orthogonal representation. Denote by $B_r \subset \mathbb{R}^n$ the ball of radius r centered at the origin. Put $d_H = \dim((\mathbb{R}^n)^H)$. Then, for each $(H) \in \Phi(G)$ and generic r ,*

$$|ch_H(Ind^G(v|_{B_r}))| = \left| \sum_{k=0}^{d_H} (-1)^k \chi(\partial_k^+ B_r^H) \right| \leq O((\mathbb{R}^n)^H; 2, m_1, \dots, m_n).$$

Proof. Take $x_1^2 + \dots + x_n^2 - r^2$ for the role of Q from Theorem 4.2 and pick r so that v is generic with respect to $\partial_1 B_r$. \square

5. THE GOTTLIEB AND GAUSS-BONNET EQUIVARIANT FORMULAS

Our next goal is to reinterpret Gottlieb's formulas ([G1]) for indices of pullback vector fields within an equivariant setting.

Let us recall the notion of a *pullback* field (see [G1]). Let $F : X \rightarrow Y$ be a differentiable map of two n -dimensional Riemannian manifolds, and w a vector field on Y . Let F^*w be a field on X defined by the formula

$$\langle (F^*w)(x), u(x) \rangle_X = \langle w, DF(u(x)) \rangle_Y$$

⁹For example, take $Q = a_1 x_1^d + \dots + a_n x_n^d \in \mathcal{P}_\Psi$ (all $a_i > 0$) plus a lower degree polynomial.

, where $x \in X$, $u(x) \in T_x X$ is a generic vector, and $\langle \sim, \sim \rangle$ denotes the scalar product in the appropriate tangent space. In other words, if ω is a 1-form dual to w in Y , then F^*w is dual to $F^*\omega$ in X .

In the case of a gradient field $w = \nabla f$ on Y ($f : Y \rightarrow \mathbb{R}$ being a smooth function), $F^*w = \nabla(f \circ F)$.

Note that when F is an equivariant map, both metrics on X and Y are G -invariant, and w is an invariant field, then F^*w is invariant as well.

We recall the notion of an equivariant degree $Deg^G(F) \in A(G)$ of a G -map $F : X \rightarrow Y$ between two compact G -manifolds of the same dimension (cf. [L]). Crudely, it is an element of $\prod_{(H) \in conj(G)} \mathbb{Z}$ whose (H) -component is the usual $deg(F^H)$, where $F^H : X^H \rightarrow Y^H$. In fact, such an element $Deg^G(F) \in A(G)$. This naive construction runs into some complications because of the ambiguities in choosing orientations of fixed point components, both in the source and the target. In a sense, one wants to coordinate the orientations of X^H and Y^H (when they are of the same dimension). The issues with the coherent orientations can be resolved by introducing some additional synchronizing structure called in [L] "the $O(G)$ -transformation of the fiber transports". Roughly speaking, it assigns a transfer G -map $(T_{F(x)}Y)^c \rightarrow (T_x X)^c$ to each $x \in X$.

Fortunately, we need to employ $Deg^G(F)$ in a particular situation, where the synchronization of the orientations can be achieved by pedestrian means. Consider an equivariant immersion f of a closed oriented $(n-1)$ -dimensional G -manifold Z into an real n -dimensional space V of an orthogonal G -representation. Denote by $S(V)$ the unit sphere centered at the origin. To each $x \in Z$ we assign the unit vector $n(x)$ tangent to V at $f(x)$ and normal to the f -image of a small neighborhood $U_x \subset Z$ of x . The orientations of V and Z help to resolve the two-fold ambiguity in picking $n(x)$.

Since f is an immersion, for any $H \subset G$, we get $Z^H = f^{-1}(f(Z) \cap V^H)$. Moreover, because $n(x)$ is orthogonal to $f(U_x)$ at $f(x) \in V^H$, $n(x)$ must be parallel to V^H . Indeed, if $n(x)$ would have a nontrivial component ν_x normal to V^H , ν_x must be moved by elements of $H \setminus \{1\}$; on the other hand, $n(x)$ is H -invariant since $T_{f(x)}(U_x)$ is.

In fact, $f^H : Z^H \rightarrow V^H$ is an immersion as well. Therefore an orientation of V^H , with the help of the "field" $n(x)$ (we assume that $x \in Z^H \setminus Sing(f^H)$), picks a particular orientation of Z^H . In the following, we assume that the orientations of V^H and Z^H are always synchronized in this way.

Thus the Gaussian map $\gamma^H : Z^H \rightarrow S(V^H)$ is well-defined for any (H) that occurs as an orbit-type of Z . Unless $dim(V^H) \leq 1$, $S(V^H)$ is connected. In such case, the degree $deg(\gamma^H)$ is defined as a sum of degrees of maps $\{\gamma_\alpha^H : Z_\alpha^H \rightarrow S(V^H)\}_\alpha$, where Z_α^H denotes a typical connected component of Z^H . When $dim(V^H) = 1$, $S(V^H) = a \amalg b$, and $deg(\gamma^H)$ is defined as the sum of degrees of the two obvious maps with the singleton targets a and b .

The theorem below is an equivariant version of the "Topological Gauss-Bonnet Theorem" from [G, page 466].

Theorem 5.1. *Let V be a real vector space of dimension n on which a compact Lie group G acts orthogonally. We assume that V admits an invariant non-vanishing*

vector field w ¹⁰. Let X a compact smooth n -dimensional G -manifold with an oriented boundary $\partial_1 X$. Consider a G -map $F : X \rightarrow V$ whose Jacobian is non-zero on $\partial_1 X$. Let $v = F^*w$ be the pullback of w under F . Denote by $\{\partial_k^+ X\}_{0 \leq k \leq n}$ the Morse stratification of X induced by v .

Then the degree of the Gauss map $\gamma : \partial_1 X \rightarrow S(V)$ with values in $A(G)$ can be computed by

$$(5.1) \quad \text{Deg}^G(\gamma) = \chi^G(X) - \text{Ind}^G(v) = - \sum_{k=1}^n (-1)^k \chi^G(\partial_k^+ X)$$

Hence, $\text{Ind}^G(v)$ and the RHS of (5.1) are w -independent¹¹.

Proof. Since $DF|_{\partial_1 X}$ is of the maximal rank, we can pullback the G -invariant riemannian metric g in V to an equivariant collar of $\partial_1 X \subset X$ and then extend the pullback $F^*(g)$ to an invariant metric on X . Let n be the unitary field outward normal to $\partial_1 X$. Then, as we described prior to the statements of Theorem 5.1, the Gauss map $\gamma : x \rightarrow DF(n(x))$, $x \in \partial_1 X$, is well-defined, equivariant, and helps to pick coherent orientations of components in $\partial_1(X^H)$. We have noticed already that $n(x)$ is contained in $T_x(X^H)$ and is normal to $\partial_1(X^H)$, provided $x \in X^H$. Also, for $x \in X^H$, $v(x) \in T_x(X^H)$. We can apply the non-equivariant Gottlieb's formula to each $\gamma^H : \partial_1(X^H) \rightarrow S(V^H)$ to conclude that $\deg(\gamma^H) = \chi(X^H) - \text{Ind}(v|_{X^H}) = - \sum_{k=1}^{\dim(V^H)} (-1)^k \chi(\partial_k^+(X^H))$. By Lemma 2.1, the latter sum is the (H) -component ch_H of $-\sum_{k=1}^n (-1)^k \chi^G(\partial_k^+(X)) \in A(G)$. Finally, $\{\deg(\gamma^H)\}_{(H) \in \Phi(G)}$ detect $\text{Deg}^G(\gamma)$. Along the way, we have shown that $\text{Deg}^G(\gamma) \in A(G)$. \square

Remark Formula (5.1) tells us that we can define $\text{Deg}^G(\gamma) \in A(G)$ as

$$(5.2) \quad - \sum_{k=1}^n (-1)^k \chi^G(\partial_k^+ X)$$

for any choice of an invariant field $w \neq 0$ in V , thus a priori avoiding all the troubles with the orientations. On the other hand, this definition of $\text{Deg}^G(\gamma)$ makes perfect sense for any equivariant immersion $f : Z \rightarrow V$ of a closed oriented G -manifold Z of codimension one: just define $\partial_0^+ Z$ to be the locus $\{x \in Z \mid \langle n(x), w(x) \rangle \leq 0\}$, and then proceed as in (1.1). We conjecture that, in general, $\sum_{k=0}^{n-1} (-1)^k \chi^G(\partial_k^+ Z)$ is w -independent. To prove this conjecture will suffice to construct an equivariant coboundary X for Z and to extend $f : Z \rightarrow V$ to a G -map $F : X \rightarrow V$.

Corollary 5.1. *Let g denotes the G -invariant Riemannian metric on V . Under notations and hypotheses of Theorem 5.1 and we get*

$$(5.3) \quad \int_{\partial_1 X^H} K_H d\mu_H = - \sum_{k=1}^{\dim(V^H)} (-1)^k \chi(\partial_k^+ X^H)$$

Here we are employing the pullback metric $F^*(g)$ in vicinity of $\partial_1 X \subset X$ to generate the volume form $d\mu_H$ on $\partial_1 X^H$ and its normal curvature K_H .

¹⁰For example, such $w \neq 0$ exists when $V = U \oplus \mathbb{R}^1$, where U is the space of a G -representation: just put $w = \nabla(f)$, where $f : U \oplus \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is the obvious projection.

¹¹Note that (5.1) proves that $\text{Deg}^G(\gamma)$, a priori an element of $\prod_{(H) \in \text{conj}(G)} \mathbb{Z}$, actually belongs to $A(G)$.

6. PROBABLE REFINEMENTS OF THE EQUIVARIANT MORSE FORMULA

One can refine the definition Birnside ring $A(G)$ in a number of ways. For example, following Dieck [D1, I.10.3], one can introduce the *component category* $\Pi_0(G, X)$ of a G -space X whose objects are G -maps $x : G/H \rightarrow X$. A morphism σ from $x : G/H \rightarrow X$ to $y : G/K \rightarrow X$ is a G -map $\sigma : G/H \rightarrow G/K$ such that x and $y \circ \sigma$ are G -homotopic. Denote by $Is \Pi_0(G, X)$ the set of isomorphism classes $[x]$ of objects $x : G/H \rightarrow X$. Let $\mathcal{A}^G(X) := \mathbb{Z}[Is \Pi_0(G, X)]$, where $\mathbb{Z}[S]$ stands for a free abelian group with the basis S . In fact, there is a bijection

$$Is \Pi_0(G, X) \rightarrow \coprod_{(H) \in conj(G)} WH \setminus \pi_0(X^H).$$

This construction defines a covariant functor $\mathcal{A}^G(\sim)$ on the category of G -spaces X with values in the category of abelian groups, a functor which is sensitive to the connected component structure of the sets $\{X^H\}$. We conjecture that all previous results can be restated in terms of $\mathcal{A}^G(\sim)$ along the lines of [LR].

However, we would like to stress a different generalization of $A(G)$ and to speculate about the corresponding Morse Formulas. In this generalization one pays a close attention to the H -representations $\{\psi_\alpha^H\}_\alpha$ arising in the normal bundles $\nu(X^H, X)$.

Let G be a compact Lie group. For each $(H) \in \Phi(G)$, fix a set $\Psi(H)$ of distinct isomorphism classes of representations $\{\psi_\alpha^H : H \rightarrow GL(V_\alpha)\}_\alpha$ with the property $V_\alpha^H = \{0\}$. Moreover, for any $\psi_\alpha \in \Psi(H)$ and $K \subset H$, the representation $\tilde{Res}_K(\psi_\alpha) : K \rightarrow GL(V_\alpha/V_\alpha^K)$ is required to be in $\Psi(K)$. We call a collection $\mathcal{F} := \{\psi_\alpha\}_{(H) \in \Phi(G), \alpha \in \Psi(H)}$ of such representations a *normal family*.

Let $X_{\psi_a}^H$ denote the set of points in X^H with the normal representations isomorphic to ψ_a .

Definition 6.1. Let $\mathcal{F} = \mathcal{F}(G)$ be a normal family of representations. Two compact smooth G -manifolds X and Y are \mathcal{F} -equivalent if, for each $(H) \in \Phi(G)$, and $\psi_a \in \Psi(H) \subset (G)$,

$$\chi(X_{\psi_a}^H) = \chi(Y_{\psi_a}^H).$$

We denote by $A(G, \mathcal{F})$ the group of such equivalence classes.

Conjecture 6.1. All the equivariant Morse formulas above are valid in the refined Burnside group $A(G, \mathcal{F})$.

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